Denver Math Club
December 2021

1. The number 1337 has the property that each of the first 3 digits are less than the last digit. How many 4 digit numbers, including 1337, have this property?

2. Let ℓ be a line with negative slope passing through the point (20,16). What is the minimum possible area of a triangle that is bounded by the x-axis, y-axis, and ℓ?

3. Compute the number of ordered quadruples of positive integers (a, b, c, d) such that $a! \cdot b! \cdot c! \cdot d! = 24!.$

4. Points E and F lie inside quadrilateral ABCD such that angle DAE is equivalent to angle EAF and angle FAB, and angle ADE is equivalent to angle EDF and angle FDC. If angle ABC is 120° and angle BCD is 90°, find angle AED plus angle AFD in degrees.

5. In triangle $ABC,$ $AB = 13,$ $BC = 14,$ and $CA = 15.$ A circle of radius r passes through point A and is tangent to line BC at $C.$ Compute r, expressing your answer as a common fraction.

6. How many integers n with $10 \leq n \leq 500$ have the property that the hundreds digit of $17n$ and $17n + 17$ are different?

7. Kelvin the frog is solving a 2021 by 2021 crossword puzzle that has exactly one black square in every column. Kelvin decides to cheat by moving the black squares. Every minute, he can move any one of the black squares either up or down 1 square. Let n be the expected number of minutes it will take Kelvin to move all the black squares to the central row. Find the sum of the digits of n.

8. The one hundred U.S. Senators are standing in a line in alphabetical order. Each senator either always tells the truth or always lies. The ith person in line says: “Of the $101 - i$ people who are not ahead of me in line (including myself), more than half of them are truth-tellers." How many possibilities are there for the set of truth-tellers on the U.S. Senate?

9. A fair 100-sided die is rolled twice, giving the numbers a and b in that order. Find the probability that $a^2 - 4b$ is a perfect square, expressing your answer as a common fraction.

10. In triangle *ABC*, we have *AB*=*AC*=20 and *BC*=14. Consider points *M* on $\overline{AB}$ and *N* on $\overline{AC}$. If the minimum value of the sum $BN + MN + MC$ is x, compute $100x$.

11. Let $N = 10^6$. For which integer a with $0 \leq a \leq N - 1$ is the value of $\binom{N}{a+1} - \binom{N}{a}$ maximized?