DMC September 2020

1. Find the prime factorization of 20^{21}, the number of divisors of 20^{21}, and the sum of the divisors of 20^{21}.

2. If a positive integer n has 7 divisors, how many divisors does n^2 have?

3. Find the remainder when 2203543267 is divided by
(a) 2, (b) 3, (c) 4, (d) 5, (e) 6, (f)8, (g) 9, (h) 10, (i) 11.

4. How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

5. Suppose x and y be positive integers such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{7}.$$

Find the sum of all possible values of x.

6. Find the number of ordered pairs of positive integers (m, n) such that $m^2 n = 20^{21}$.

7. What two-digit number is three times the sum of its digits?

8. In how many consecutive zeroes does the product $115 \times 116 \times 117 \times \ldots \times 201$ end?

9. What four-digit number has tens digit 2 and units digit 8, is a multiple of 16, and when its digits are reversed the result is also a multiple of 16?

10. Find the sum of the prime factors of $2^{12} - 1$.

11. Aiden writes his favorite four-digit positive integer on a piece of paper. Brandon reverses Aiden's number. If the resulting number is four times Aiden's number, find Aiden's number.

12. Find the remainder when 20^{21} is divided by (a) 7 and (b) 77.

13. Kevin writes a sequence of numbers starting with 1, and repeatedly adding 1 until a multiple of 2 is reached. He then repeatedly adds 2 to this value until a multiple of 3 is reached, then adds 3 until he gets a multiple of 4, and so on. The first four terms are $1, 2, 4, 6$. What will be the last term Kevin writes down before he adds 2000 for the first time?

14. What is the greatest integer less than or equal to

$$\frac{3^{100} + 2^{100}}{3^{96} + 2^{96}}?$$

15. When 9! is expressed as an integer in base 9, the result ends in m zeros, and the last nonzero digit immediately preceding the m zeros is n. What is the value of the ordered pair (m, n)?

16. Let m and n be positive integers satisfying the conditions $\gcd(m+n, 210) = 1$, m^m is a multiple of n^n, and m is not a multiple of n. Find the least possible value of $m + n$.

Answer key:

1. $2^{42} \cdot 5^{21}$, 946, $(2^{43} - 1)(5^{22} - 1)$

2. 13

3. (a) 1, (b) 1, (c) 3, (d) 2, (e) 1, (f) 3, (g) 7, (h) 7, (g) 9

4. 100

5. 78

6. 242

7. 27

8. 23

10. 28

11. 2178

12. (a) 6, (b) 20

13. 3998000

14. 81

15.$(2, 7)$

16. 407