Number Theory

Denver Math Club November 2021

1. Malcolm wants to visit Isabella after school today and knows the street where she lives but doesn't know her house number. She tells him, ”My house number has two digits, and exactly three of the following four statements about it are true.” (1) It is prime. (2) It is even. (3) It is divisible by 7. (4) One of its digits is 9. What is Isabella's house number?

2. The least common multiple of a and b is 12, and the least common mul- tiple of b and c is 15. What is the least possible value of the least common multiple of a and c?

3. What is the largest integer n for which 5^n is a factor of the sum 98!+99!+100! ?

4. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

5. The product (8)(888 . . . 8), where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k?

6. Let N = 123456789101112 . . . 4344 be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

7. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her grandchildren?

8. Cozy the Cat and Dash the Dog are going up a staircase with a certain number of steps. However, instead of walking up the steps one at a time, both Cozy and Dash jump. Cozy goes two steps up with each jump (though if nec- essary, he will just jump the last step). Dash goes five steps up with each jump (though if necessary, he will just jump the last steps if there are fewer than 5 steps left). Suppose the Dash takes 19 fewer jumps than Cozy to reach the top of the staircase. Let s denote the sum of all possible numbers of steps this staircase can have. What is the sum of the digits of s?

9. A rectangular box measures $a \times b \times c$, where a, b, and c are integers and $1 \leq a \leq b \leq c$. The volume and surface area of the box are numerically equal. How many ordered triples (a, b, c) are possible?

10. Find the number of 10-digit numbers which are multiples of 11 such that the digits are non-increasing from left to right.