July DMC Solutions

July 2017

1 Part 1

1. Since $9 = 3^2$, we can convert each digit in base 9 into a two digit base 3 number, and then concatenate them to get the answer. So, we get $2_9 = 02_3, 6_9 = 20_3, 4_9 = 11_3, 7_9 = 21_3, 8_9 = 22_3, 1_9 = 01_3$, and our answer is 1220221112002_3

2. Consider 2 factors of n, d and $\frac{n}{d}$, other than 1 and n. Notice how $d \cdot \frac{n}{d} = n$, and thus we can't have any other factors or else the product of the distinct proper divisors would be greater than n. Hence, n can only have 4 factors, so is either of the form p^3 or $p \cdot q$ for distinct primes p, q. The smallest such numbers are $2 \cdot 3 = 6, 2^3 = 8, 2 \cdot 5 = 10, 2 \cdot 7 = 14, 3 \cdot 5 = 15, 3 \cdot 7 = 21, 2 \cdot 11 = 22, 2 \cdot 13 = 26, 3^3 = 27, 3 \cdot 11 = 33$, and adding we get 182.

3. We see $(3, 2) * (0, 0) = (3, 2)$, and this must equal $(x, y) * (3, 2) = (x-3, y+2)$, so we need $x - 3 = 3$, and thus $x = 6$.

4. Let's consider the sequence modulo 3, which is to say look at the remainder of each term when divided by 3. We have $F_1 = 1, F_2 = 0, F_3 = 1, F_4 = 0, \ldots$ and so on, with $F_n = 1$ if n is odd and $F_n = 0$ if n is even. Thus, F_n is divisible by 3 if and only if n is even, so our answer is $100/2 = 50$.

5. We factor out 12! to get $12!(1 + 13 \cdot 14) = 12! \cdot 183 = 12! \cdot 3 \cdot 61$, and since all prime factors of 12 factorial are less than 12, our answer is 61.

6. Using the clock-angle formula, which says the angle is $|30h - 5.5m|$ where h is the hour and m is the minute, we get $|30 \cdot 4 - 5.5 \cdot 14 = |120 - 77| = 43$.

7. Since each team has a 1/2 chance of winning each game, they are equally matched and our answer, by symmetry, is $\frac{1}{2}$.

8. Since $2|n! + 2, 3|n! + 3, 4|n! + 4, \ldots n - 1|n! + n - 1$, we see that they are all composite, so our answer is 0.

9. We use mass points. WLOG $AE = CE = 1$, so we assign a weight of 1 to both A and C. Then, WLOG $CD = 2, BD = 1$, so B has weight 2.

Also, E's weight is the sum of A and C's weights, so it is 2. Then, since the weights of B and E are equal, $BF = FE$. Now, $\frac{[BDF]}{[FDCE]} = \frac{[BDF]}{[BCE]-[BDF]}$. Then, $[BCE] = \frac{BE}{BF} \cdot \frac{BC}{BD} \cdot [BDF]$, so we get the ratio is $\frac{[BDF]}{6[BDF]-[BDF]} = \frac{1}{5}$.

10. Using the 3-D distance formula, we get $PD = \sqrt{5^2 + \frac{9^2}{5^2} + \frac{12^2}{5^2}} = \sqrt{34}$. Alternatively, note that DA is orthogonal to plane ABP, and thus we get $PD = \sqrt{AD^2 + AP^2} = \sqrt{34}$.

11. Working backwards, we get $f(f(k)) = 54$, since $27 - 3 = 24$ is not odd. Then, $f(k) = 108$ or 51, and $k = 105, 216$, or 102. Of these, only 105 is odd, and our answer is $1 + 0 + 5 = 6$.

12. Let $n = 2^a \cdot 3^b \cdot k$, where k is not a multiple of 2 or 3. Then, if k has k_1 factors, we get $2n$ has $(a + 2)(b + 1)k_1 = 28$ factors, and $3n$ has $(a + 1)(b + 2)k_1 = 30$ factors. Since k_1 is a factor of 28 and 30 it is either 1 or 2. If it is 1, then $(a+2)(b+1) = 28, (a+1)(b+2) = 30$, which has solution $(a, b) = (5, 3)$, and $k_1 = 2$ renders no solutions. Thus, $6n = 2^{a+1} \cdot 3^{b+1} \cdot k$, and has $(a + 2)(b + 2)k_1 = 7 \cdot 5 \cdot 1 = 35$ factors.

13. Multiplying the first equation by 9 and subtracting away the second, we get $9x^2 - (x - 4)^2 = 144$. This gives $x^2 + x - 20 = 0$, meaning $x = 4, -5$, giving solutions as $(x, y) = (4, 3), (4, -3), (-5, 0)$ and the area of this triangle is 27.

14. Setting them equal, and canceling the x^2, we get $ax + b = cx + d(a - c)x = d - b$ has at least one solution. Complementary counting, that doesn't have a solution iff $a = c$ and $d \neq b$, which happens with probability $\frac{1}{6} \cdot \frac{5}{6}$, so our answer is $1 - \frac{5}{36} = \frac{31}{36}$.

15. Let $\angle ABD = \angle CBD = x$, and since $\angle ABD = \angle ACB$ by similarity, $\angle ACB = x$ as well, so since the angles in a triangle add to 180, we get $117 + 3x = 180x = 21$, and $\angle ABC = 2x = 42$.

16. Clearly if one of the digits is 0, then the others all have to be 0, contradiction. Now, notice that if we pick any 4 distinct integers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, there is a unique way to rearrange them in a four-digit integer such that each digit is smaller than the digit to it's left, and so our answer is just $94 = 210$.

17. Suppose there are only bicycles. With 28 seats, this would give $28 * 2 = 56$ wheels, but we have 67, or 11 extra, wheels. Every time we switch out a tricycle for a bicycle, we increase the number of wheels by one while the number of seats remain constant, so we need to switch out 11 bicycles to get 67 wheels, and thus there are 11 tricycles.

18. A variant of Chicken McNugget Theorem gives $\frac{(3-1)(7-1)}{2} = 6$ as our answer (alternatively you can just list).

19. We have $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$, so we check the maximum power of each of these dividing 213!. Legendre's formula gives the number of 2s in 213 as $\lfloor 2132 \rfloor + \lfloor 2134 \rfloor + \lfloor 2138 \rfloor + \lfloor 21316 \rfloor + \lfloor 21332 \rfloor + \lfloor 21364 \rfloor + \lfloor 213128 \rfloor = 208$, but since the exponent of 2 is 7, the number of 2^7s in 213! is $\lfloor 2087 \rfloor = 29$. Checking $3, 5, 7$ in pretty much the same way, we get higher numbers, and thus 29 is our answer.

20. We factorize to get $10920 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13$, so to make a perfect square, we need to make all the exponents even. Thus, our answer is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$.

21. In order to be a triangle, the third side must be at least 4, and since $14^2 + 4^2 < 17^2$, we get that this triangle is obtuse and our answer is 4.

22. We have $x^2 + y^2 + z^2 = 446, xyz = 1638$, and $xy + yz + zx = 425$. Now, $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+xz) = 1296$, so $x+y+z = 36$. The volume of the new prism is $(x+2)(y+2)(z+2) = xyz+2xy+2xz+2yz+4x+4y+4z+8 = 1638 + 850 + 144 + 8 = 2640$.

23. Notice that this means the numerator and denominator of the product are equal, so if we multiply the numerator and denominator, we must get a perfect square. However, the product of the numerator and denominator is the product of all the terms, namely $((n-1)!)^2 \cdot n$, so n must be a perfect square. In order to show n being a perfect square works, consider the product with $\frac{k}{k-1}$ reciprocated when $k \geq \sqrt{n}$, and not-reciprocated otherwise. We see that this indeed works, so our answer is the sum of the first four perfect squares greater than 1, which is $4 + 9 + 16 + 25 = 54$.

24. WLOG $EF = 4$. Letting K be the midpoint of EH, we see $EC = CK = 1$, and $AC = \sqrt{4-1} = \sqrt{3} = BD$, so our answer is, since $CD = EF$, $\frac{4-2\sqrt{3}}{4} = \frac{2-\sqrt{3}}{2}$.

25. If the three primes are odd, so is their sum, and thus one of the primes must be even, which is to say it equals 2. Then, we need to maximize $2pq$ with p, q primes adding to 98. In order to maximize their product, we want to make them as close as possible. We see this occurs when $p = 37, q = 61$, and thus our answer is $2 \cdot 37 \cdot 61 = 4514$.

26. We use casework. If there are 0 heads, we get 1 possible order of flips. With 1 heads, we get 5 possible orders. With 2 heads, we get $52 - 4 = 6$ possibilities, where we subtract away the cases when the heads are consecutive. Finally, with three heads, the order must be $HTHTH$, for 1 case, and our answer is $\frac{1+5+6+1}{2^5} = \frac{13}{32}$.

27. First, notice that, by symmetry, if n^2 has k factors, then $\frac{k-1}{2}$ of them

are less than n. In this case, $k = 63 * 38 = 2457$, so $\frac{k-1}{2} = 1228$. Then, since it cannot divide n, we must subtract the factors of n less than n (since any factor of n is a factor of n^2). There are $32 * 20 - 1 = 639$ factors of n less than n, so our answer is $1228 - 639 = 589$.

28. The square can intersect each of the circles in 8 distinct places, and the circles can intersect each other twice, so our answer is $8 \cdot 2 + 2 = 18$.

29. Suppose the train reaches the tunnel in k seconds. Then it will reach the end of the tunnel in $\frac{\frac{3}{5}}{\frac{2}{5}}k = \frac{3k}{2}$ seconds. That means it can go through the tunnel in $\frac{k}{2}$ seconds, while the man can go through the tunnel in $\frac{5k}{2}$ seconds, so the train is 5 times as fast as the man, and our answer is $60/5 = 12$.

30. Note that $1.75 + \sqrt{3} = 1 + \frac{3}{4} + \sqrt{3} = (1 + \sqrt{3}2)^2$! This means the expression is $2 + \sqrt{3} - (1.75 + \sqrt{3}) = .25$.

2 Part 2

1. We see 168 clips equals $168 \cdot \frac{8}{7} = 192$ claps, and 192 claps equals $192 \cdot \frac{5}{12} = 80$ clups, so our answer is 80.

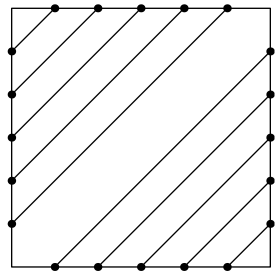
2. In the worst case scenario, we will get one color for as long as possible, which entails picking 94 red marbles. After this, we are guaranteed another color, so our answer is 95.

3. We get, letting the width be w, that $w \cdot (3w - 44) = 119$. Since $119 = 7 \cdot 17$, we see $w = 17$ works, and that is our answer.

4. NOTE: The question should read what is the greatest lower bound of $b^2 + a^2$, or the largest number which $b^2 + a^2$ can't equal. Plugging into the formula, we get the equation becomes $a^2b - a^2 - a = a^2b - a^2 - ab + aab = 2a$, and thus either $a = 0$, but they are nonzero, or $b = 2$. Then, making a infinitesimally small, we get the answer is 4.

5. Note that the units digits of powers of 3 go $3, 9, 7, 1, 3, 9, 7, 1, ...$, so we want to find $3^{3^3} \mod 4$. This is $3^{27} \equiv (-1)^{27} \equiv -1 \equiv 3 \pmod 4$, and so we pick the third number in the sequence to get 7.

6. We see that the minimum is 0, as we can connect the dots to form a sequence of almost, if not exactly, parallel lines that are nearly parallel to the diagonal, like so:

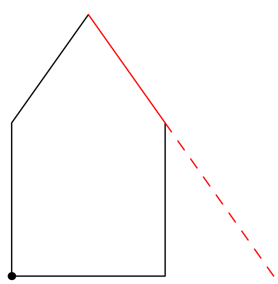


Now, the best case scenario is when we connect each point to a point opposite it such that any two lines intersect. There are $5 \cdot 4/2 = 10$ lines, and hence the maximum is $102 = 45$ when each line intersects every other line. Thus, our answer is $45 - 0 = 45$.

7. If p is odd, then $p^4 + 1$ is even and a multiple of 2, so isn't prime as it can't equal 2. Thus, $p = 2$, and the answer is $2^4 + 1 = 17$.

8.We complementary count. If it reaches $(2, 2)$, then it must go up,up,right,right in some order. There are $4!/(2 \cdot 2) = 6$ ways to arrange those directions, and the probabilities of them happening are $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$, so the probability the caterpillar reaches $(2, 2)$ is $\frac{27}{128}$, and the answer is $1 - \frac{27}{128} = \frac{101}{128}$.

9. Placing the die on the square face opposite the pyramid, we see the slope of the pyramid's slanted faces is $-\sqrt{2}$, using pythagorean theorem. Now, the highest corner of the die upon rolling it and landing on a triangular face is equal in height to all points on the highest edge, so taking a cross section slicing the die in two congruent parts and passing through the midpoint of the highest edge, we get the following shape:



We want to find the length of the perpendicular from the bottom left vertex to the red line (which has slope $-\sqrt{2}$). The equation of the red line is, letting the dot be the origin, $y - 1 - \sqrt{2} + x\sqrt{2} = 0$. Thus, using the point to line distance formula, our answer is $\frac{|1+\sqrt{2}|}{\sqrt{1+2}} = \frac{\sqrt{3}+\sqrt{6}}{3}$.

10. Since any combination of dimes and nickels must be a multiple of 5, must be a multiple of 5, if there are $7p$ pennies, then $2018 - 7p$ must be a multiple of 5, so $p \equiv 4, 9 \pmod{10}$. Now, doing casework on p, we get the number of

possibilities is $200+196+193+189+186+182+...$ where we alternate between -4 and -3. Taking this as the sum of two arithmetic sequences and combining them into one, namely $(200+193+186+...+4)+(196+189+182+...+7+0) = 396+382+368+...+18+4 = \frac{29(396+4)}{2} = 5800$, we get our answer.